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Unification of Spins and Charges[1]

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Polynomials in Grassmann algebra can be used to describe the internal degrees, spins and charges of spinors, scalars, and vectors. It was shown by Mankoč Borštnik and Nielsen that Kähler spinors can be generalized to describe spins of vectors as well as spins and charges of scalars, vectors, and spinors. In dimensions 14 and higher, the spontaneous breaking of symmetry leads gravity in *d* dimensions to manifest in 4-dimensional subspace as ordinary gravity and all needed gauge fields as well as the Yukawa couplings. Both approaches, Kähler's one (if generalized) and ours, manifest four generations of massless fermions, which are left-handed *SU*(2) doublets and right-handed *SU*(2) singlets. A possible way of spontaneously breaking symmetries is pointed out at the level of canonical momentum.

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1. INTRODUCTION

We have shown (Mankoč Borštnik, 1992a, b, 1993, 1994a–c, 1995a, b, 1999; Mankoč Borštnik and Fajfer, 1997; Mankoč Borštnik and Mankoč Borštnik, 1999) how a space of anticommuting coordinates can be used to unify spins and charges and that gravity in *d*-dimensional space manifests, after a breaking of symmetry, in 4-dimensional subspace as ordinary gravity and all known gauge fields. Kähler (1962) showed how to use differential forms to describe the spin of fermions. We comment on the necessity of the appearance of four copies of Dirac fermions in both approaches. Comparing the two approaches, we generalized the Kähler approach to describe also integer spins as well as charges for either spinors or vectors, unifying spins and charges (Mankoč Borštnik and Nielsen, 1999).

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We present the possible Lagrange function for a free particle and the canonical quantization of anticommuting coordinates (Mankoč Borštnik, 1992a, b, 1993, 1994a–c, 1995a, b, 1999). Introducing vielbeins and spin connections, we demonstrate how the spontaneous breaking of symmetry may lead to the symmetries of the Standard Model. We show how the breaking of symmetries from $SO(1,13)$ to symmetries of the Standard Model manifests on the canonical momentum (Borštnik and Mankoč Borštnik, 1999). We show how the symmetry of the group $SO(1,13)$ breaks to $SO(1,7)$ [leading to multiplets with left-handed $SU(2)$ doublets and right-handed $SU(2)$ singlets] and $SO(6)$, which then leads to the $SO(1,3) \times SU(2) \times U(1) \times SU(3) \times U(1)$. The two $U(1)$ symmetries enable, besides the hypercharge needed in the Standard Model, an additional hypercharge which is nonzero for a right-handed $SU(2)$ and $SU(3)$ singlet, like a right-handed neutrino in the Standard Model.

2. DIRAC EQUATIONS

What we call quantum mechanics in Grassmann space is the model for going beyond the Standard Model with extra dimensions of ordinary and anticommuting coordinates describing spins and charges (Mankoč and Borštnik, 1992 a, b, 1993, 1994 a–c, 1995a, b, 1999). In a d-dimensional space-time, the internal degrees of freedom come from the Grassmann odd variables θ^a, $a \in \{0, \ldots, d\}$. The wave function describing either spinors or vectors is

$$\langle \theta^a | \Phi \rangle = \sum_{i=0,\ldots,d} \sum_{\{a_1 < a_2 < \ldots < a_i\} \in \{0,\ldots,d\}} \alpha_{a_1, a_2, \ldots, a_i} \theta^{a_1} \theta^{a_2} \cdots \theta^{a_i} \tag{1}$$

where the coefficients $\alpha_{a_1, a_2, \ldots, a_i}$ depend on commuting coordinates x^a. The wave function space spanned over Grassmannian coordinate space has the dimension 2^d. The operator for the conjugate variable θ^a is

$$p^\theta_a = -i\vec{\partial}_a$$

The right arrow indicates that the derivation has to be performed from the left-hand side. These operators then obey the odd Heisenberg algebra,

$$\{A, B\} := AB - (-1)^{n_{AB}} BA \tag{2}$$

$$n_{AB} = \begin{cases} +1 & \text{if } A \text{ and } B \text{ have Grassmann odd character} \\ 0 & \text{otherwise} \end{cases}$$

$$\{p^{\theta a}, p^{\theta b}\} = 0 = \{\theta^a, \theta^b\}, \qquad \{p^{\theta a}, \theta^b\} = -i\eta^{ab},$$

$$\eta = diag\,\{1, -1, -1, \ldots\}$$

The following operators fulfill the Clifford algebra:

$$\tilde{a}^a := i(p^{\theta a} - i\theta^a), \qquad \tilde{\tilde{a}}^a := -(p^{\theta a} + i\theta^a) \tag{3}$$

$$\{\tilde{a}^a, \tilde{a}^b\} = 2\eta^{ab} = \{\tilde{\tilde{a}}^a, \tilde{\tilde{a}}^b\}, \qquad \{\tilde{a}^a, \tilde{\tilde{a}}^b\} = 0 \tag{4}$$

The Dirac-like equations are

$$\text{either} \quad \tilde{a}^a p_a|\Phi\rangle = 0 \quad \text{or} \quad \tilde{\tilde{a}}^a p_a|\Phi\rangle = 0, \quad p_a = i\,\frac{\partial}{\partial x^a} \tag{5}$$

Applying either the operator $\tilde{a}^a p_a$ on the left-hand side equation or $\tilde{\tilde{a}}^a p_a$ on the right-hand side equation, we get the Klein–Gordon equation $p^a p_a|\Phi\rangle = 0$.

Neither of the two equations (5) has solutions which would transform as spinors with respect to the generators of the Lorentz transformations when taken in analogy with the generators of the Lorentz transformations in ordinary space,

$$L^{ab} = x^a p^b - x^b p^a, \qquad S^{ab} := \theta^a p^{\theta b} - \theta^b p^{\theta a} \tag{6}$$

$$\begin{aligned} S^{ab} &= \tilde{S}^{ab} + \tilde{\tilde{S}}^{ab}, \qquad \tilde{S}^{ab} := -\frac{i}{4}[\tilde{a}^a, \tilde{a}^b], \\ \tilde{\tilde{S}}^{ab} &:= -\frac{i}{4}[\tilde{\tilde{a}}^a, \tilde{\tilde{a}}^b], \qquad [A, B] := AB - BA \end{aligned} \tag{7}$$

The solutions of the two Dirac equations (5) transform as spinors with respect to either $\tilde{S}^{ab}$ or $\tilde{\tilde{S}}^{ab}$.

The untilded, the single-tilded, and the double-tilded S^{ab} obey the d-dimensional Lorentz generator algebra $\{M^{ab}, M^{cd}\} = -i(M^{ad}\eta^{bc} + M^{bc}\eta^{ad} - M^{ac}\eta^{bd} - M^{bd}\eta^{ac})$ when inserted in the form M^{ab}.

Kähler (1962) formulated spinors in terms of the wave function (1) which, when replacing θ^a by $dx^a \wedge$, is a superposition of the differential p-forms.

In Mankoč Borštnik and Nielsen (1999), we presented the generalization of the Kähler approach. In both approaches, two types of operators fulfilling the Clifford algebra [the ones of our approach are presented in Eqs. (3) and (4)] as well as the two Dirac-like equations [Eqs. (5) represent our Dirac-like equation] can be obtained. Both approaches offer the generators of the Lorentz transformations (6) describing not only spinors, but also vectors. In both approaches, the γ^a matrices fulfilling the Clifford algebra and having Grassmann even grade can be defined,

$$\tilde{\gamma}^a = i\tilde{\tilde{a}}^0\tilde{a}^a, \qquad \{\tilde{\gamma}^a, \tilde{\gamma}^b\} = 2\eta^{ab} \tag{8}$$

The "naive" definition of gamma matrices $\gamma^a_{\text{naive}} := \tilde{a}^a$, which change the Grassmann character of spinors, differs from the Grassmann-even definition of gamma matrices presented in Eq. (8), which keeps the Grassmann character of spinors (both fulfilling the Clifford algebra) only when γ^0-matrix has to simulate the parity reflection, which is $\vec{\theta} \to -\vec{\theta}$. In all physical applications (such as construction of currents), the two definitions cannot be distinguished, since the γ^a always appear in pairs. We can check that the $\tilde{\gamma}^a$ of (8) indeed perform the operation of parity reflection.

2.1. Scalar Product

We define the scalar product as follows (Mankoč Borštnik, 1992a, b, 1993, 1994a–c, 19995a, b, 1999):

$$\langle \Phi_1 | \Phi_2 \rangle = \int d^d\, \theta(\omega \langle \theta^a | \Phi_1 \rangle) \langle \theta^a | \Phi_2 \rangle$$

where ω is a weight function,

$$\omega = \prod_{i=0,1,\ldots,d} (\theta^i + \vec{\partial}^i)$$

which operates on only the first function $\langle \theta^a | \Phi_1 \rangle$ and

$$\int d\theta^a = 0, \qquad \int d^d\theta\, \theta^0\theta^1 \ldots \theta^d = 1, \qquad d^d\theta = d\theta^d \ldots d\theta^1\, d\theta^0$$

It follows that

$$\langle \Phi^{(1)} | \Phi^{(2)} \rangle = \sum_{0,d} \sum_{\alpha_1 < \alpha_2 < \cdots < \alpha_2} \alpha^{(1)*}_{\alpha_1 \cdots \alpha_i}\, \alpha^{(2)}_{\alpha_1 \cdots \alpha_i} \tag{9}$$

in complete analogy with the usual definition of the scalar product in ordinary space and with the Kähler definition of the scalar product of two p-forms (Kähler, 1962).

2.2. Four copies of Weyl Bispinors

In Table I, the eigenvalues of $\tilde{S}^{12}$, $\tilde{S}^{03}$, and $\tilde{\Gamma}^{(4)} := i\tilde{a}^0\tilde{a}^1\tilde{a}^2\tilde{a}^3$ are written. The Roman numerals indicate the family number. We use the relation $\tilde{a}^a|0\rangle = \theta^a$.

We present in Table I for $d = 4$ the 2^d vectors, which we arrange into four copies of two Weyl spinors, one left-handed, $\langle \tilde{\Gamma}^{(4)} \rangle = -1$, $\Gamma^{(4)} = i[(-2i)^2/4!]\epsilon_{abcd}\, \mathcal{S}^{ab}\mathcal{S}^{cd}$, $(\mathcal{S}^{ab} = \tilde{S}^{ab})$, and one right-handed, $\langle \tilde{\Gamma}^{(4)} \rangle = 1$. We have made a choice of ($\tilde{\ }$) operators, putting the operators of the type ($\tilde{\tilde{\ }}$) equal to zero. We present these vectors as polynomials of θ^m, $m \in (0, 1, 2, 3)$. The two Weyl vectors of one copy of the Weyl bispinors are connected by the $\tilde{\gamma}^m$ [Eq. (8)] operators, while the two copies of different Grassmann character are connected by $\tilde{a}^a$. The two copies of an even Grassmann character are connected by a kind of a time reversal operation $\theta^0 \to -\theta^0$.

Table I. The Polynomials of θ^m Representing the Four by Two Weyl Spinors

a	i	$\langle\theta\vert\Phi^a_i\rangle$	$\tilde{S}^{12}$	$\tilde{S}^{03}$	$\tilde{\Gamma}^{(4)}$	Family	Grade
1	1	$\frac{1}{2}\,(\tilde{a}^1 - i\tilde{a}^2)(\tilde{a}^0 - \tilde{a}^3)$	$\frac{1}{2}$	$-\frac{i}{2}$	-1		
1	2	$-\frac{1}{2}\,(1 + i\tilde{a}^1\tilde{a}^2)(1 - \tilde{a}^0\tilde{a}^3)$	$-\frac{1}{2}$	$\frac{i}{2}$	-1	I	Even
2	1	$\frac{1}{2}(\tilde{a}^1 - i\tilde{a}^2)(\tilde{a}^0 + \tilde{a}^3)$	$\frac{1}{2}$	$\frac{i}{2}$	1		
2	2	$-\frac{1}{2}(1 + i\tilde{a}^1\tilde{a}^2)(1 + \tilde{a}^0\tilde{a}^3)$	$-\frac{1}{2}$	$-\frac{i}{2}$	1		
3	1	$\frac{1}{2}\,(\tilde{a}^1 - i\tilde{a}^2)(1 - \tilde{a}^0\tilde{a}^3)$	$\frac{1}{2}$	$\frac{i}{2}$	1		
3	2	$-\frac{1}{2}\,(1 + i\tilde{a}^1\tilde{a}^2)(\tilde{a}^0 - \tilde{a}^3)$	$-\frac{1}{2}$	$-\frac{i}{2}$	1	II	Odd
4	1	$\frac{1}{2}\,(\tilde{a}^1 - i\tilde{a}^2)(1 + \tilde{a}^0\tilde{a}^3)$	$\frac{1}{2}$	$-\frac{i}{2}$	-1		
4	2	$-\frac{1}{2}(1 + i\tilde{a}^1\tilde{a}^2)(\tilde{a}^0 + \tilde{a}^3)$	$-\frac{1}{2}$	$\frac{i}{2}$	-1		
5	1	$\frac{1}{2}(1 - i\tilde{a}^1\tilde{a}^2)(\tilde{a}^0 - \tilde{a}^3)$	$\frac{1}{2}$	$-\frac{i}{2}$	-1		
5	2	$-\frac{1}{2}(\tilde{a}^1 + i\tilde{a}^2)(1 - \tilde{a}^0\tilde{a}^3)$	$-\frac{1}{2}$	$\frac{i}{2}$	-1	III	Odd
6	1	$\frac{1}{2}(1 - i\tilde{a}^1\tilde{a}^2)(\tilde{a}^0 + \tilde{a}^3)$	$\frac{1}{2}$	$\frac{i}{2}$	1		
6	2	$-\frac{1}{2}(\tilde{a}^1 + i\tilde{a}^2)(1 + \tilde{a}^0\tilde{a}^3)$	$-\frac{1}{2}$	$-\frac{i}{2}$	1		
7	1	$\frac{1}{2}(1 - i\tilde{a}^1\tilde{a}^2)(1 - \tilde{a}^0\tilde{a}^3)$	$\frac{1}{2}$	$\frac{i}{2}$	1		
7	2	$-\frac{1}{2}(\tilde{a}^1 + i\tilde{a}^2)(\tilde{a}^0 - \tilde{a}^3)$	$-\frac{1}{2}$	$-\frac{i}{2}$	1	IV	Even
8	1	$\frac{1}{2}(1 - i\tilde{a}^1\tilde{a}^2)(1 + \tilde{a}^0\tilde{a}^3)$	$\frac{1}{2}$	$-\frac{i}{2}$	-1		
8	2	$-\frac{1}{2}(\tilde{a}^1 + i\tilde{a}^2)(\tilde{a}^0 + \tilde{a}^3)$	$-\frac{1}{2}$	$\frac{i}{2}$	-1		

In Table I, four copies of the Weyl two-spinors are given as polynomials of θ^a. Eigenstates are orthonormalized according to the scalar product of Eq. (9).

Analyzing the irreducible representations of the group $SO(1, 3)$ with respect to the generator of the Lorentz transformations of the vectorial type (Mankoč Borštnik, 1993) [Eqs. (6)], one finds for $d = 4$ a scalar and a pseudoscalar, two three-vectors [in the complex version of the $SU(2) \times SU(2)$ representation of $SO(1, 3)$ denoted by (1, 0) and (0, 1) representation, respectively, with $\langle\Gamma^{(4)}\rangle = \pm 1$], and two four-vectors. The polynomial representation for this case is given in Mankoč Borštnik (1993).

2.3. Generalization to Extra Dimensions

It has been suggested (Mankoč Borštnik, 1994) that the Lorentz transformations in the space of θ^a in $(d - 4)$ dimensions manifest themselves as generators for charges observable for the four-dimensional particles. The extra dimensional spin degrees and the ordinary spin degrees originate from the θ^a and we have a unification of these internal degrees of freedom.

Let us take as an example the model (Mankoč Borštnik, 1995 ; Borštnik and Mankoč Borštnik, 1999) which has $d = 14$ and at first, at the high-energy level, $SO(1, 13)$ Lorentz group, but which should be broken in two steps to first $SO(1, 7) \times SO(6)$ and then to $SO(1, 3) \times SU(3) \times SU(2)$. See Section 5.

2.4. Appearance of Spinors

One wonders how it is at all possible that the Dirac equation appears for a spinor field out of models with only scalar, vector, and tensor objects. It only can be done by exchanging the Lorentz generators $\mathcal{S}^{ab}$ by the $\tilde{S}^{ab}$, say (or the $\tilde{\tilde{S}}^{ab}$ if we choose them instead) of Eq. (7). This indeed means that one of the two kinds of operators fulfilling the Clifford algebra—a choice of $\tilde{\tilde{a}}^a$ having been made—are put to zero in the operators of the Lorentz transformations as well as in all the operators representing physical quantities. The use of $\tilde{a}^0$ in the operator $\tilde{\gamma}^0$ (and equivalently also in the Dirac case) is the exception, only used to simulate the Grassmann even parity operation $\vec{\theta} \to -\vec{\theta}$.

We shall argue away (Mankoč Borštnik, 1994a–c, 1995a, b, 1999) the $\tilde{\tilde{a}}^a$ in Section 3 on the ground of the action.

3. LAGRANGE FUNCTION FOR FREE MASSLESS PARTICLES AND CANONICAL QUANTIZATION

We present in this section the Lagrange function for a particle which lives in $X^a \equiv \{x^a, \theta^a\}$ and has its geodesics parametrized by a Grassmann even parameter τ and a Grassmann odd parameter ξ, $\xi^2 = 0$. The coordinates $X^a = X^a(x^a, \theta^a, \tau, \xi)$ are called the supercoordinates. We define the dynamics of a particle by choosing the action (Ikemori, 1987; Mankoč Borštnik, 1992a, b, 1993, 1994 a–c, 1995a, b, 1999)

$$I = \frac{1}{2}\int d\tau\, d\xi\, E E^i_A\, \partial_i X^a E^j_B\, \partial_j X^b \eta_{ab}\eta^{AB}, \qquad \partial_i := (\partial_\tau, \vec{\partial}_\xi), \quad \tau^i = (\tau, \xi)$$

while E^i_A determines a metric on a two-dimensional superspace τ^i, $E = det(E^i_A)$. We choose $\eta_{AA} = 0$, $\eta_{12} = 1 = \eta_{21}$, while η_{ab} is the Minkowski metric with the diagonal elements $(1, -1, -1, -1, \ldots, -1)$. The action is invariant under the Lorentz transformations of supercoordinates: $X'^a = \Lambda^a{}_b X^b$. Since a supermatrix $E^i{}_A$ transforms as a vector in a two-dimensional superspace τ^i under general coordinate transformations of τ^i, $E^i{}_A \tau_i$ is invariant under such transformations and so is $d^2\tau\, E$. The action is locally supersymmetric. The inverse matrix $E^A{}_i$ is defined as follow: $E^i{}_A E^B{}_i = \delta^B{}_A$.

Either x^a or θ^a depends on time parameter τ and $\xi^2 = 0$, therefore the geodesics can be described as follows: $X^a = x^a + \epsilon\xi\theta^a$. We choose ϵ^2 to be equal either to $+i$ or to $-i$ so that it defines two possible combinations of supercoordinates. We choose the metric $E^i{}_A$: $E^1{}_1 = 1$, $E^1{}_2 = -\varepsilon M$, $E^2{}_1 = \xi$, $E^2{}_2 = N - \varepsilon\xi M$, with N and M Grassmann even and odd parameters, respectively. We write $\dot{A} = (d/d\tau)A$ for any A.

If we integrate the above action over the Grassmann odd coordinate $d\xi$, the action for a superparticle follows:

$$\int d\tau \left(\frac{1}{N}\dot{x}^a\dot{x}_a + \varepsilon^2\dot{\theta}^a\theta_a - \frac{2\varepsilon^2 M}{N}\dot{x}^a\theta_a\right) \tag{10}$$

Defining the two momenta

$$p^\theta_a := \frac{\overrightarrow{\partial} L}{\partial\dot{\theta}^a} = \epsilon^2\theta^a, \qquad p_a := \frac{\partial\, L}{\partial\dot{x}^a} = \frac{2}{N}(\dot{x}_a - Mp^{\theta a}) \tag{11}$$

the two Euler–Lagrange equations follow:

$$\frac{dp^a}{d\tau} = 0, \qquad \frac{dp^{\theta a}}{d\tau} = \varepsilon^2\,\frac{M}{2}\,p^a \tag{12}$$

Variation of the action [Eq. (10)] with respect to M and N gives the two constraints

$$\chi^1 := p^a a^\theta_a = 0, \qquad \chi^2 = p^a p_a = 0, \qquad a^\theta_a := ip^\theta_a + \varepsilon^2\theta_a \tag{13}$$

while $\chi^3{}_a := -p^\theta_a + \epsilon^2\theta_a = 0$ [Eq. (11)] is the third type of constraint of the action (10). For $\varepsilon^2 = -i$, we find that $a^\theta{}_a = \tilde{a}^a$, which agrees with Eq. (3), while $\chi^3{}_a = \tilde{\tilde{a}}_a = 0$, which makes a choice between $\tilde{a}^a$ and $\tilde{\tilde{a}}^a$.

We find the generators of the Lorentz transformations for the action (10) to be

$$M^{ab} = L^{ab} + S^{ab}, \qquad L^{ab} = x^a\,p^b - x^b p^a,$$
$$S^{ab} = \theta^a p^{\theta b} - \theta^b p^{\theta a} = \tilde{S}^{ab} + \tilde{\tilde{S}}^{ab}$$

which agree with definitions in Eq. (7) and show that parameters of the Lorentz transformations are the same in both spaces.

The Hamilton function and the Poisson brackets are

$$H := \dot{x}^a p_a + \dot{\theta}^a p^\theta{}_a - L = \tfrac{1}{4}Np^a p_a + \tfrac{1}{2}Mp^a(\tilde{a}_a + i\tilde{\tilde{a}}_a)$$
$$\{A, B\}_p = \frac{\partial A}{\partial x^a}\frac{\partial B}{\partial p_a} - \frac{\partial A}{\partial p_a}\frac{\partial B}{\partial x^a} + \frac{\overrightarrow{\partial A}}{\partial\theta^a}\frac{\overrightarrow{\partial B}}{\partial p^\theta_a} + \frac{\overrightarrow{\partial A}}{\partial p^\theta_a}\frac{\overrightarrow{\partial B}}{\partial\theta^a} \tag{14}$$

The Poisson brackets fulfill the algebra of the generalized commutators (Mankoč Borštnik, 1992a, b, 1993, 1994a–c, 1995a, b, 1999) of Eq. (2).

If we take into account the constraint $\chi^3{}_a = \tilde{\tilde{a}}_a = 0$ in the Hamilton function (which just means that instead of H, the Hamilton function. $H + \Sigma_i \alpha^i \chi^i + \Sigma_a \alpha^3{}_a \chi^{3a}$ is taken, with parameters α^i, $i = 1, 2$, and $\alpha^3{}_a = -\frac{1}{2} p_a$, $a = 0, 1, 2, 3, 5, \ldots, d$, chosen in such a way that the Poisson brackets of the three types of constraints with the new Hamilton function are equal to zero) and in all dynamical quantities, we find

$$H = \tfrac{1}{4} N p^a p_a + \tfrac{1}{2} M p^a \tilde{a}_a, \qquad \chi^1 = p^a p_a = 0, \qquad \chi^2 = p^a \tilde{a}_a = 0$$
$$\dot{p}_a = \{p_a, H\}_P = 0, \qquad \dot{\tilde{a}}_a = \{\tilde{a}_a, H\}_P = iMp_a$$

which agrees with the Euler–Lagrange equations (12). We further find

$$\dot{\chi}^i = \{H, \chi^i\}_P = 0, \quad i = 1, 2, \qquad \dot{\chi}^3{}_a = \{H, \chi^3{}_a\}_P = 0, \quad a = 0, \ldots, d$$

which guarantees that the three constraints will not change with the time parameter τ and that $\dot{\tilde{M}}^{ab} = 0$, with $\tilde{M}^{ab} = L^{ab} + \tilde{S}^{ab}$, saying that $\tilde{M}^{ab}$ is the constant of motion.

The Dirac brackets, which can be obtained from the Poisson brackets of Eq. (14) by adding to these brackets on the right-hand side a term $-\{A, \tilde{\tilde{a}}^c\}_P \cdot (-(1/2i)\eta_{ce}) \cdot \{\dot{\tilde{a}}^e, B\}_P$, give for the dynamical quantities, which are observables, the same results as the Poisson brackets. This is true also for $\tilde{a}^a$, $(\{\tilde{a}^a, \tilde{a}^b\}_D = i\eta^{ab} = \{\tilde{a}^a, \tilde{a}^b\}_P)$, which is the dynamical quantity, but not an observable since its odd Grassmann character causes supersymmetric transformations. We also find that $\{\tilde{a}^a, \tilde{\tilde{a}}^b\}_D = 0 = \{\tilde{a}^a, \tilde{\tilde{a}}^b\}_P$. The Dirac brackets give different results only for the quantities θ^a and $p^{\theta a}$ and for $\tilde{\tilde{a}}^a$ among themselves: $\{\theta^a, p^{\theta b}\}_P = \eta^{ab}$, $\{\theta^a, p^{\theta b}\}_D = \frac{1}{2}\eta^{ab}$, $\{\tilde{\tilde{a}}^a, \tilde{\tilde{a}}^b\}_P = 2i\eta^{ab}$, $\{\tilde{\tilde{a}}^a, \tilde{\tilde{a}}^b\}_D = 0$. According to the above properties of the Poisson brackets, we suggested (Mankoč Borštnik, 1995 , 1999) that in the quantization procedure the Poisson brackets (14) rather than the Dirac brackets be used, so that variables $\tilde{\tilde{a}}^a$, which are removed from all dynamical quantities, stay as operators. Then $\tilde{a}^a$, and $\tilde{\tilde{a}}^a$ are expressible with θ^a and $p^{\theta a}$ [Eq. (3)] and the algebra of linear operators introduced in Section 2 can be used. We shall show that the suggested quantization procedure leads to the Dirac equation, which is the differential equation in ordinary and Grassmann space and has all the desired properties.

In the quantization procedure $-i\{A, B\}_p$ goes to either a commutator or to an anticommutator according to the Poisson brackets (14). The operators θ^a, $p^{\theta a}$ (in the coordinate representation, they become $\theta^a \to \theta^a p^\theta_a \to i\vec{\partial}/\partial\theta^a$) fulfill the Grassmann odd Heisenberg algebra, while the operators $\tilde{a}^a$ and $\tilde{\tilde{a}}^a$ fulfill the Clifford algebra [Eq. (4)].

The constraints (13) lead to the Weyl-like and the Klein–Gordon equations

$$p^a \tilde{a}_a |\tilde{\Phi}\rangle = 0, \qquad p^a p_a |\Phi\rangle = 0 \qquad \text{with} \quad p^a \tilde{a}_a p^b \tilde{a}_b = p^a p_a$$

Trying to solve the eigenvalue problem $\tilde{\tilde{a}}^a|\tilde{\Phi}\rangle = 0$, $a = (0, \ldots, d)$, we find that no solution of this eigenvalue problem exists, which means that the third constraint $\tilde{\tilde{a}} = 0$ cannot be fulfilled in the operator form (although we take it into account in the operators for all dynamical variables in order for the operator equations to agree with the classical equations). We can only take it into account in the expectation value form

$$\langle\tilde{\Phi}|\tilde{\tilde{a}}^a|\tilde{\Phi}\rangle = 0 \tag{15}$$

Since $\tilde{\tilde{a}}^a$ are Grassmann odd operators, they change monomials [Eq. (1)] of a Grassmann odd character into monomials of an Grassmann even character and vice versa, which is the supersymmetry transformation. It means that Eq. (15) is fulfilled for monomials of either odd or even Grassmann character and that superpositions of the Grassmann odd and Grassmann even monomials are not solutions for this system.

Let

$$\tilde{\Upsilon} = i^a \prod_{a=0,1,2,3,5,\ldots,d} \tilde{a}^a \sqrt{\eta^{aa}}, \qquad \tilde{\tilde{\Upsilon}} = i^\alpha \prod_{a=0,1,2,3,5,\ldots,d} \tilde{\tilde{a}}^a \sqrt{\eta^{aa}}$$

$$(\tilde{\Upsilon})^2 = 1 = (\tilde{\tilde{\Upsilon}})^2, \qquad P_\pm = \tfrac{1}{2}\left(1 \pm \sqrt{(-)^{\tilde{\Upsilon}\tilde{\tilde{\Upsilon}}}}\,\tilde{\Upsilon}\tilde{\tilde{\Upsilon}}\right), \qquad (P_\pm)^2 = P_\pm \tag{16}$$

with α equal either to $d/2$ or to $(d-1)/2$ for even and odd dimension d of the space, respectively.

We can use the projector $P_\pm$ of Eq. (16) to project out of monomials either the Grassmann odd or the Grassmann even part. Since this projector commutes with the Hamilton function ($\{P_\pm, H\} = 0$), it means that eigenfunctions of H, which fulfill Eq. (15), have either an odd or an even Grassmann character. In order that, in the second quantization procedure, the fields $|\tilde{\Phi}\rangle$ describe fermions, we accept in the fermion case Grassmann odd monomials only.

4. PARTICLES IN GAUGE FIELDS

The dynamics of a point particle in gauge fields, the gravitational field in d dimensions, which then, as we shall show, manifests in the subspace $d = 4$ as ordinary gravity and all the Yang–Mills fields, can be obtained by transforming vectors from a freely falling to an external coordinate system (Wess and Bagger, 1983). To do this, supervielbeins $\mathbf{e}^a{}_\mu$ have to be introduced, which in our case, depend on ordinary and on Grassmann coordinates as well as on two types of parameters $\boldsymbol{\tau}^i = (\tau, \xi)$. The index a refers to a freely falling coordinate system (a Lorentz index), the index μ refers to an external coordinate system (an Einstein index).

The transformation of vectors is $\partial_i X^a = \mathbf{e}^a{}_\mu \partial_i X^\mu$, $\partial_i X^\mu = \mathbf{f}^\mu{}_a \partial_i X^a$, $\partial_i = (\partial_\tau, \partial_\xi)$. It follows that $\mathbf{e}^a{}_\mu \mathbf{f}^\mu{}_b = \delta^a{}_b$, $\mathbf{f}^\mu{}_a \mathbf{e}^a{}_\nu = \delta^\mu{}_\nu$.

A Taylor expansion of vielbeins with respect to ξ is $\mathbf{e}^a{}_\mu = e^a{}_\mu + \varepsilon^2 \xi \theta^b e^a{}_{\mu b}$, $\mathbf{f}^\mu{}_a = f^\mu{}_a - \varepsilon^2 \xi \theta^b f^\mu{}_{ab}$.

Both expansion coefficients again depend on ordinary and on Grassmann coordinates. Having an even Grassmann character $e^a{}_\mu$ will describe the spin-2 part of a gravitational field. The coefficients $e^a{}_{\mu b}$ define the spin connections (Mankoč Borštnik, 1992a, b, 1995a–c, 1999). It follows that $e^a{}_\mu f^\mu{}_b = \delta^a{}_b$, $f^\mu{}_a e^a{}_\nu = \delta^\mu{}_\nu$, $e^a{}_{\mu b} f^\mu{}_c = e^a{}_\mu f^\mu{}_{cb}$.

We find the metric tensor $\mathbf{g}_{\mu\nu} = \mathbf{e}^a{}_\mu \mathbf{e}_{a\nu}$, $\mathbf{g}^{\mu\nu} = \mathbf{f}^\mu{}_a \mathbf{f}^{\nu a}$. Rewriting the action from Section 3 in terms of an external coordinate system, using the Taylor expansion of supercoordinates X^μ and superfields $\mathbf{e}^a{}_\mu$, and integrating the action over the Grassmann odd parameter ξ, the action follows:

$$I = \int d\tau \left\{ \frac{1}{N} g^{\mu\nu} \dot{x}^\mu \dot{x}^\nu - \varepsilon^2 \frac{2M}{N} \theta_a e^a{}_\mu \dot{x}^\mu + \varepsilon^2 \frac{1}{2} (\dot{\theta}^\mu \theta_a - \theta_a \dot{\theta}^\mu) e^a{}_\mu \right.$$
$$\left. + \varepsilon^2 \frac{1}{2} (\theta^b \theta_a - \theta_a \theta^b)\, e^a{}_{\mu b} \dot{x}^\mu \right\}$$

which defines the two momenta of the system $p_\mu = \partial L / \partial \dot{x}^\mu = p_{0\mu} + \frac{1}{2} \tilde{S}^{ab} e_{a\mu b}$ and $p^\theta_\mu = -i\theta_a e^a{}_\mu (\varepsilon^2 = -i)$. Here $p_{0\mu}$ are the canonical (covariant) momenta of a particle. For $p^\theta_a = p^\theta_\mu f^\mu{}_a$, it follows that p^θ_a is proportional to θ_a. Then $\tilde{a}_a = i(p^\theta_a - i\theta_a)$, while $\tilde{\tilde{a}}_a = 0$. We may further write

$$p_{0\mu} = p_\mu - \tfrac{1}{2} \tilde{S}^{ab} e_{a\mu b} = p_\mu - \tfrac{1}{2} \tilde{S}^{ab} \omega_{ab\mu}, \qquad \omega_{ab\mu} = \tfrac{1}{2}(e_{a\mu b} - e_{b\mu a}) \tag{17}$$

which is the usual expression for the covariant momenta in gauge gravitational fields (Wess and Bagger, 1983) with two constraints

$$p^\mu_0 p_{0\mu} = 0 = p_{0\mu} f^\mu{}_a \tilde{a}^a \tag{18}$$

5. BREAKING *SO*(1,13) THROUGH *SO*(1,7) × *SO*(6) TO *SO*(1,3) × *SO*(2) × *U*(1) × *SU*(3)

The breaking of symmetries leads in 4-dimensional subspace to ordinary gravity and all the gauge fields.

In this section, we first discuss a possible breaking of symmetry which leads from the unified theory of only spins and gravity in d dimensions to spins and charges and to the symmetries and assumptions of the Standard Model, on the algebraic level in Section 5.1. We then comment on the breaking of symmetries on the level of canonical momentum, first for the Standard

Model case (Section 5.2.1), and then for the particle in the presence of the gravitational field (Section 5.2.2).

We shall present as well the possible explanation for that postulate of the Standard Model which requires that only left-handed, weak charged massless doublets and right-handed, weak charged massless singlets exist, and accordingly connect spins and charges of fermions.

5.1. Algebraic Considerations of Symmetries

The algebra of the group $SO(1, d-1)$ or $SO(d)$ contains n subalgebras defined by operators τ^{Ai} ($A = 1, \ldots, n$; $i = 1, \ldots, n_A$;) where n_A is the number of elements of each subalgebra) with the properties (Mankoč Borštnik, 1995 ; Borštnik and Mankoč Borštnik; 1999, Mankoč Borštnik, and Fajfer, 1997)

$$[\tau^{Ai}, \tau^{Bj}] = i\delta^{AB} f^{Aijk} \tau^{Ak} \tag{19}$$

if operators τ^{Ai} can be expressed as linear superpositions of operators M^{ab}

$$\tau^{Ai} = c^{Ai}{}_{ab} M^{ab}, \qquad c^{Ai}{}_{ab} = -c^{Ai}{}_{ba},$$
$$A = 1, \ldots, n, \quad i = 1, \ldots, n_A, \quad a, b = 1, \ldots, d$$

Here f^{Aijk} are structure constants of the A subgroup with n_A operators. According to the three kinds of operators $\mathcal{S}^{ab}$, two of spinorial and one of vectorial character, there are three kinds of operators τ^{Ai} defining subalgebras of spinorial and vectorial character, respectively, those of spinorial types being expressed with either $\tilde{S}^{ab}$ or $\tilde{\tilde{S}}^{ab}$ and those of vectorial type being expressed by S^{ab}. All three kinds of operators are, according to Eq. (19), defined by the same coefficients $c^{Ai}{}_{ab}$ and the same structure constants f^{Aijk}. From Eq. (19), the following relations among constants $c^{Ai}{}_{ab}$ follow:

$$-4c^{Ai}{}_{ab} c^{Bjb}{}_{c} - \delta^{AB} f^{Aijk} c^{Ak}{}_{ac} = 0$$

When we look for coefficients $c^{Ai}{}_{ab}$ which express operators τ^{Ai}, forming a subalgebra $SU(n)$ of an algebra $SO(2n)$ in terms of M^{ab}, the procedure is rather simple (Georgi, 1982; Mankoč Borštnik, 1997),

$$\tau^{Am} = -\frac{i}{2} (\tilde{\sigma}^{Am})_{jk} \{M^{(2j-1)(2k-1)} + M^{(2j)(2k)} + iM^{(2j)(2k-1)} - iM^{(2j-1)(2k)}\}$$

Here $(\sigma^{Am})_{jk}$ are the traceless matrices which form the algebra of $SU(n)$. One can easily prove that operators τ^{Am} fulfill the algebra of the group $SU(n)$ for any of three choices for operators M^{ab}: S^{ab}, $\tilde{S}^{ab}$, $\tilde{\tilde{S}}^{ab}$.

While the coefficients are the same for all three kinds of operators, the representations depend on the operators M^{ab}. After solving the eigenvalue problem for invariants of subgroups, the representations can be presented as polynomials of coordinates θ^a, or $dx^a\wedge$, $a = 0, 1, 2, 3, 5, \ldots, 14$. The operators of spinorial character define the fundamental representations of the group and the subgroups, while the operators of vectorial character define the adjoint representations of the groups. We shall from now on, for the sake of simplicity, refer to the polynomials of Grassmann coordinates only.

We first analyze the space of 2^d vectors for $d = 14$ with respect to commuting operators (Casimirs) of subgroups $SO(1, 7)$ and $SO(6)$, so that polynomials of θ^0, θ^1, θ^2, θ^3, θ^5, θ^6, θ^7, and θ^8 are used to describe states of the group $SO(1, 7)$ and then polynomials of θ^9, θ^{10}, θ^{11}, θ^{12}, θ^{13}, and θ^{14} further describe states of the group $SO(6)$. The group $SO(1, 13)$ has the rank equal to $r = 7$ since it has 7 commuting operators (namely, for example, $\mathcal{S}^{01}$, $\mathcal{S}^{23}$, $\mathcal{S}^{56}, \ldots, \mathcal{S}^{13\,14}$), while the ranks of the subgroups $SO(1, 7)$ and $SO(6)$ are accordingly $r = 4$ and $r = 3$, respectively. We may further decide to arrange the basic states in the space of polynomials of $\theta^0, \ldots, \theta^8$ as eigenstates of four Casimirs of the subgroups $SO(1, 3)$, $SU(2)$, and $U(1)$ (the first has $r = 2$, the second and the third have $r = 1$) of the group $SO(1, 7)$, and the basic states in the space of polynomials of $\theta^9, \ldots, \theta^{14}$ as eigenstates of $r = 3$ Casimirs of subgroups $SU(3)$ and $U(1)$ (with $r = 2$ and $r = 1$, respectively) of the group $SO(6)$.

We presented in Table I the eight Weyl spinors, two by two—one left-handed ($\tilde{\Gamma}^{(4)} = -1$) and one right-handed ($\tilde{\Gamma}^{(4)} = 1$)—connected by $\tilde{\gamma}^n$, $n = 0, 1, 2, 3$, into Weyl bispinors. The two four-vectors of the same grade are connected by the discrete time reversal operation $\theta^0 \to -\theta^0$ (Mankoč Borštnik and Nielsen, 1999), while the two four-vectors, which differ in Grassmann character, are connected by the operation of $\tilde{a}^a$.

According to Eq. (19), one can express the generators of the subgroups $SU(2)$ and $U(1)$ of the group $SO(1, 7)$ in terms of the generators $\mathcal{S}^{ab}$. We find

$$\tau^{38} := \tfrac{1}{2}(\mathcal{S}^{58} - \mathcal{S}^{67}), \qquad \tau^{32} := \tfrac{1}{2}(\mathcal{S}^{57} + \mathcal{S}^{68}), \qquad \tau^{33} := \tfrac{1}{2}(\mathcal{S}^{56} - \mathcal{S}^{78})$$
$$\tau^{41} := \tfrac{1}{2}(\mathcal{S}^{56} + \mathcal{S}^{78})$$

The algebra of Eq. (19) follows since the operators τ^{Ai} have an even Grassmann character, and the generalized commutation relations agree with the usual commutators, denoted by [,],

$$\{\tau^{3i}, \tau^{3j}\} = i\epsilon_{ijk}\tau^{3k}, \qquad \{\tau^{41}, \tau^{3i}\} = 0$$

One notices that $\tau^{51} := \frac{1}{2}(\mathcal{S}^{58} + \mathcal{S}^{67})$ and $\tau^{52} := \frac{1}{2}(\mathcal{S}^{57} - \mathcal{S}^{68})$ together with τ^{41} form the algebra of the group $SU(2)$ and that the generators of this group commute with τ^{3i}.

We present in Table II the eigenvectors of the operators $\tilde{\tau}^{33}$ and $(\tilde{\tau}^3)^2 = (\tilde{\tau}^{31})^2 + (\tilde{\tau}^{32})^2 + (\tilde{\tau}^{33})^2$, which are at the same time the eigenvectors of $\tilde{\tau}^{41}$

Table II. The Eigenstates of $\tilde{\tau}^{33}$, $\tilde{\tau}^{41}$

a	i	$\langle\theta\|\Phi^{a}{}_{i}\rangle$	$\tilde{\tau}^{33}$	$\tilde{\tau}^{41}$	Grade
1	1	$\frac{1}{2}(1 - i\tilde{a}^5\tilde{a}^6)(1 + i\tilde{a}^7\tilde{a}^8)$	$-\frac{1}{2}$	0	
1	2	$-\frac{1}{2}(\tilde{a}^5 + i\tilde{a}^6)(\tilde{a}^7 - i\tilde{a}^8)$	$\frac{1}{2}$	0	
2	1	$\frac{1}{2}(1 + i\tilde{a}^5\tilde{a}^6)(1 - i\tilde{a}^7\tilde{a}^8)$	$\frac{1}{2}$	0	
2	2	$-\frac{1}{2}(\tilde{a}^5 - i\tilde{a}^6)(\tilde{a}^7 + i\tilde{a}^8)$	$-\frac{1}{2}$	0	
					Even
3	1	$\frac{1}{2}(1 + i\tilde{a}^5\tilde{a}^6)(1 + i\tilde{a}^7\tilde{a}^8)$	0	$\frac{1}{2}$	
4	1	$\frac{1}{2}(\tilde{a}^5 + i\tilde{a}^6)(\tilde{a}^7 + i\tilde{a}^8)$	0	$\frac{1}{2}$	
5	1	$\frac{1}{2}(1 - i\tilde{a}^5\tilde{a}^6)(1 - i\tilde{a}^7\tilde{a}^8)$	0	$-\frac{1}{2}$	
6	1	$\frac{1}{2}(\tilde{a}^5 - i\tilde{a}^6)(\tilde{a}^7 - i\tilde{a}^8)$	0	$-\frac{1}{2}$	
7	1	$\frac{1}{2}(1 + i\tilde{a}^5\tilde{a}^6)(\tilde{a}^7 - i\tilde{a}^8)$	$\frac{1}{2}$	0	
7	2	$-\frac{1}{2}(i\tilde{a}^5 - i\tilde{a}^6)(1 + \tilde{a}^7\tilde{a}^8)$	$-\frac{1}{2}$	0	
8	1	$\frac{1}{2}(1 - i\tilde{a}^5\tilde{a}^6)(\tilde{a}^7 + i\tilde{a}^8)$	$-\frac{1}{2}$	0	
8	2	$-\frac{1}{2}(\tilde{a}^5 + i\tilde{a}^6)(1 - i\tilde{a}^7\tilde{a}^8)$	$\frac{1}{2}$	0	
					Odd
9	1	$\frac{1}{2}(1 - i\tilde{a}^5\tilde{a}^6)(\tilde{a}^7 - i\tilde{a}^8)$	-0	$-\frac{1}{2}$	
10	1	$\frac{1}{2}(\tilde{a}^5 + i\tilde{a}^6)(1 + \tilde{a}^7\tilde{a}^8)$	0	$\frac{1}{2}$	
11	1	$\frac{1}{2}(1 + i\tilde{a}^5\tilde{a}^6)(\tilde{a}^7 + i\tilde{a}^8)$	0	$\frac{1}{2}$	
12	1	$\frac{1}{2}(\tilde{a}^5 - i\tilde{a}^6)(1 - \tilde{a}^7\tilde{a}^8)$	0	$-\frac{1}{2}$	

for spinors. We find, with respect to the group $SU(2)$, two doublets and four singlets of an even and another two doublets and four singlets of an odd Grassmann character.

In Table II, we have two doublets and four singlets of an even Grassmann character and two doublets and four singlets of an odd Grassmann character. One sees that complex conjugation transforms one doublet of either odd or even Grassmann character into another of the same Grassmann character, changing the sign of the value of $\tilde{\tau}^{33}$, while it transforms one singlet into another singlet of the same Grassmann character and of the opposite value of $\tilde{\tau}^{41}$. One can check that $\tilde{a}^h$, $h \in (5, 6, 7, 8)$, transforms the doublets of an even Grassmann character into singlets of an odd Grassmann character.

One sees that $\tilde{\tau}^{5i}$, $i = 1, 2$, transform doublets into singlets [which can easily be understood by taking into account that $\tilde{\tau}^{5i}$ close together with τ^{41} the algebra of $SU(2)$ and that the two $SU(2)$ groups are isomorphic to the group $SO(4)$].

One also sees the following very important property of representations of the group $SO(1, 7)$: Applying the operators $\tilde{S}^{ab}$, $a, b = 0, 1, 2, 3, 5, 6, 7, 8$, on the direct product of polynomials of Tables I and II, which form the representations of the group $SO(l, 7)$, one finds that a multiplet of $SO(1, 7)$ exists which contains left-handed $SU(2)$ doublets and right-handed $SU(2)$

singlets. There exists also another multiplet which contains left-handed $SU(2)$ singlets and right-handed $SU(2)$ doublets. It turns out that the operators $\tilde{S}^{mh}$, with $m = 0, 1, 2, 3$ and $h = 5, 6, 7, 8$, although having an even Grassmann character, change the Grassmann character of that part of the polynomials which belong to Tables I and II, respectively, keeping the Grassmann character of the products of the two types of polynomials unchanged. This can be understood by taking into account that $\tilde{S}^{mh} = -(i/2)\tilde{a}^m\tilde{a}^h$ and that the operator $\tilde{a}^m$ changes the polynomials of an odd Grassmann character of Table I into an even polynomial, transforming a left-handed Weyl spinor of one family into a right-handed Weyl spinor of another family, while $\tilde{a}^h$ changes simultaneously the $SU(2)$ doublet of an even Grassmann character into a singlet of an odd Grassmann character.

The symmetry, called the mirror symmetry, presented in this approach is not broken, as none of the symmetry is broken. We only have arranged basic states to demonstrate possible symmetries.

We can express the generators of subgroups $SU(3)$ and $U(1)$ of the group $SO(6)$ in terms of the generators $\mathcal{S}^{ab}$ [according to Eq. (19)]. The indices 9, 10, 11, 12, 13, and 14 are reserved for the subgroup $SO(6)$,

$$\tau^{61} := \tfrac{1}{2}(\mathcal{S}^{9\,12} - \mathcal{S}^{10\,11}), \qquad \tau^{62} := \tfrac{1}{2}(\mathcal{S}^{9\,11} + \mathcal{S}^{10\,12}),$$
$$\tau^{63} := \tfrac{1}{2}(\mathcal{S}^{9\,10} - \mathcal{S}^{11\,12})$$
$$\tau^{64} := \tfrac{1}{2}(\mathcal{S}^{9\,14} - \mathcal{S}^{10\,13}), \qquad \tau^{65} := \tfrac{1}{2}(\mathcal{S}^{9\,13} + \mathcal{S}^{10\,14}),$$
$$\tau^{66} := \tfrac{1}{2}(\mathcal{S}^{11\,14} - \mathcal{S}^{12\,13})$$
$$\tau^{67} := \tfrac{1}{2}(\mathcal{S}^{11\,13} + \mathcal{S}^{12\,14}), \qquad \tau^{68} := \frac{1}{2\sqrt{3}}(\mathcal{S}^{9\,10} + \mathcal{S}^{11\,12} - 2\mathcal{S}^{13\,14})$$
$$\tau^{71} := -\tfrac{1}{3}(\mathcal{S}^{9\,10} + \mathcal{S}^{11\,12} + \mathcal{S}^{13\,14})$$

The algebra for the subgroups $SU(3)$ and $U(1)$ follows from the algebra of the Lorentz group $SO(1, 13)$,

$$\{\tau^{6i}, \tau^{6j}\} = if_{ijk}\tau^{6k}, \qquad \{\tau^{71}, \tau^{6i}\} = 0 \qquad \text{for each} \quad i$$

The coefficients f_{ijk} are the structure constants of the group $SU(3)$.

We can find the eigenvectors of the Casimirs of the groups $SU(3)$ and $U(1)$ for spinors as polynomials of θ^h, $h = 9, \ldots, 14$. The eigenvectors, which are polynomials of an even grade, are given in (Mankoč Borštnik and Fajfer, 1997). Tables III and IV give polynomials of an odd grade (Borštnik and Mankoč Borštnik, 1999).

In Tables III and IV one sees that complex conjugation transforms one triplet into an antitriplet, while $\tilde{\tau}^{8i}$ transform triplets into antitriplets or singlets.

Table III. The Eigenstates of $\tilde{\tau}^{63}$, $\tilde{\tau}^{68}$ of the Group $SU(3)$, $\tilde{\tau}^{71}$ [$U(1)$], for Polynomials of the Odd Grade: Four Triplets and Four Antitriplets

a	i	$\sqrt{2^3}\,\langle\theta\vert\Phi^a{}_i\rangle$	$2\tilde{\tau}^{63}$	$\sqrt{3}\tilde{\tau}^{68}$	$\tilde{\tau}^{71}$
1	1	$(1 + i\tilde{a}^{13}\tilde{a}^{14})(\tilde{a}^9 - i\tilde{a}^{10})(1 + i\tilde{a}^{11}\tilde{a}^{12})$	1	$\frac{1}{2}$	$\frac{1}{6}$
1	2	$(1 + i\tilde{a}^{13}\tilde{a}^{14})(1 + i\tilde{a}^9\tilde{a}^{10})(\tilde{a}^{11} - i\tilde{a}^{12})$	-1	$\frac{1}{2}$	$\frac{1}{6}$
1	3	$-(\tilde{a}^{13} - i\tilde{a}^{14})(1 + i\tilde{a}^9\tilde{a}^{10})(1 + i\tilde{a}^{11}\tilde{a}^{12})$	0	-1	$\frac{1}{6}$
2	1	$(1 + i\tilde{a}^{13}\tilde{a}^{14})(1 - i\tilde{a}^9\tilde{a}^{10})(\tilde{a}^{11} + i\tilde{a}^{12})$	1	$\frac{1}{2}$	$\frac{1}{6}$
2	2	$(1 + i\tilde{a}^{13}\tilde{a}^{14})(\tilde{a}^9 + i\tilde{a}^{10})(1 - i\tilde{a}^{11}\tilde{a}^{12})$	-1	$\frac{1}{2}$	$\frac{1}{6}$
2	3	$-(\tilde{a}^{13} - i\tilde{a}^{14})(i\tilde{a}^9 + i\tilde{a}^{10})(\tilde{a}^{11} + i\tilde{a}^{12})$	0	-1	$\frac{1}{6}$
3	1	$(\tilde{a}^{13} + i\tilde{a}^{14})(i\tilde{a}^9 - i\tilde{a}^{10})(\tilde{a}^{11} + i\tilde{a}^{12})$	1	$\frac{1}{2}$	$\frac{1}{6}$
3	2	$(\tilde{a}^{13} + i\tilde{a}^{14})(1 + i\tilde{a}^9\tilde{a}^{10})(1 - i\tilde{a}^{11}\tilde{a}^{12})$	-1	$\frac{1}{2}$	$\frac{1}{6}$
3	3	$-(1 - i\tilde{a}^{13}\tilde{a}^{14})(1 + i\tilde{a}^9\tilde{a}^{10})(\tilde{a}^{11} + i\tilde{a}^{12})$	0	-1	$\frac{1}{6}$
4	1	$(\tilde{a}^{13} + i\tilde{a}^{14})(1 - i\tilde{a}^9\tilde{a}^{10})(1 + i\tilde{a}^{11}\tilde{a}^{12})$	1	$\frac{1}{2}$	$\frac{1}{6}$
4	2	$(\tilde{a}^{13} + i\tilde{a}^{14})(\tilde{a}^9 + i\tilde{a}^{10})(\tilde{a}^{11} - i\tilde{a}^{12})$	-1	$\frac{1}{2}$	$\frac{1}{6}$
4	3	$-(1 - i\tilde{a}^{13}\tilde{a}^{14})(\tilde{a}^9 + i\tilde{a}^{10})(1 + i\tilde{a}^{11}\tilde{a}^{12})$	0	-1	$\frac{1}{6}$
5	1	$(1 - i\tilde{a}^{13}\tilde{a}^{14})(\tilde{a}^9 + i\tilde{a}^{10})(1 - i\tilde{a}^{11}\tilde{a}^{12})$	-1	$-\frac{1}{2}$	$-\frac{1}{6}$
5	2	$(1 - i\tilde{a}^{13}\tilde{a}^{14})(1 - i\tilde{a}^9\tilde{a}^{10})(\tilde{a}^{11} + i\tilde{a}^{12})$	1	$-\frac{1}{2}$	$-\frac{1}{6}$
5	3	$-(\tilde{a}^{13} + i\tilde{a}^{14})(1 - i\tilde{a}^9\tilde{a}^{10})(1 - i\tilde{a}^{11}\tilde{a}^{12})$	0	1	$-\frac{1}{6}$
6	1	$(1 - i\tilde{a}^{13}\tilde{a}^{14})(1 + i\tilde{a}^9\tilde{a}^{10})(\tilde{a}^{11} - i\tilde{a}^{12})$	-1	$-\frac{1}{2}$	$-\frac{1}{6}$
6	2	$(1 - i\tilde{a}^{13}\tilde{a}^{14})(\tilde{a}^9 - i\tilde{a}^{10})(1 + i\tilde{a}^{11}\tilde{a}^{12})$	1	$-\frac{1}{2}$	$-\frac{1}{6}$
6	3	$-(\tilde{a}^{13} + i\tilde{a}^{14})(i\tilde{a}^9 - i\tilde{a}^{10})(\tilde{a}^{11} - i\tilde{a}^{12})$	0	1	$-\frac{1}{6}$
7	1	$(\tilde{a}^{13} - i\tilde{a}^{14})(i\tilde{a}^9 + i\tilde{a}^{10})(\tilde{a}^{11} - i\tilde{a}^{12})$	-1	$-\frac{1}{2}$	$-\frac{1}{6}$
7	2	$-(\tilde{a}^{13} - i\tilde{a}^{14})(1 - i\tilde{a}^9\tilde{a}^{10})(1 + i\tilde{a}^{11}\tilde{a}^{12})$	1	$-\frac{1}{2}$	$-\frac{1}{6}$
7	3	$-(1 + i\tilde{a}^{13}\tilde{a}^{14})(1 - +i\tilde{a}^9\tilde{a}^{10})(\tilde{a}^{11} - i\tilde{a}^{12})$	0	1	$-\frac{1}{6}$
8	1	$(\tilde{a}^{13} - i\tilde{a}^{14})(1 + i\tilde{a}^9\tilde{a}^{10})(1 - i\tilde{a}^{11}\tilde{a}^{12})$	-1	$-\frac{1}{2}$	$-\frac{1}{6}$
8	2	$(\tilde{a}^{13} - i\tilde{a}^{14})(\tilde{a}^9 - i\tilde{a}^{10})(\tilde{a}^{11} + i\tilde{a}^{12})$	1	$-\frac{1}{2}$	$-\frac{1}{6}$
8	3	$-(1 + i\tilde{a}^{13}\tilde{a}^{14})(\tilde{a}^9 - i\tilde{a}^{10})(1 - i\tilde{a}^{11}\tilde{a}^{12})$	0	1	$-\frac{1}{6}$

Table IV. The Eigenstates as Polynomials of the Odd Grade: Eight Singlets with $\tilde{\tau}^{63} = 0$ and $\tilde{\tau}^{68} = 0$

$\sqrt{2^3}\,\langle\theta\vert\Phi^a\rangle$	$2\tilde{\tau}^{71}$
$(1 + i\tilde{a}^{13}\tilde{a}^{14})(\tilde{a}^9 + i\tilde{a}_{10})(1 + i\tilde{a}^{11}\tilde{a}^{12})$	1
$(1 + i\tilde{a}^{13}\tilde{a}^{14})(1 + i\tilde{a}^9\tilde{a}^{10})(\tilde{a}^{11} + i\tilde{a}^{12})$	1
$(\tilde{a}^{13} + i\tilde{a}^{14})(1 + i\tilde{a}^9\tilde{a}^{10})(1 + i\tilde{a}^{11}\tilde{a}^{12})$	1
$(\tilde{a}^{13} + i\tilde{a}^{14})(\tilde{a}^9 + i\tilde{a}^{10})(\tilde{a}^{11} + i\tilde{a}^{12})$	1
$(1 - i\tilde{a}^{13}\tilde{a}^{14})(\tilde{a}^9 - i\tilde{a}^{10})(1 - i\tilde{a}^{11}\tilde{a}^{12})$	-1
$(1 - i\tilde{a}^{13}\tilde{a}^{14})(1 - i\tilde{a}^9\tilde{a}^{10})(\tilde{a}^{11} - i\tilde{a}^{12})$	-1
$(\tilde{a}^{13} - i\tilde{a}^{14})(1 - i\tilde{a}^9\tilde{a}^{10})(1 - i\tilde{a}^{11}\tilde{a}^{12})$	-1
$(\tilde{a}^{13} - i\tilde{a}^{14})(\tilde{a}^9 - i\tilde{a}^{10})(\tilde{a}^{11} - i\tilde{a}^{12})$	-1

The following operators transform triplets of the group $SU(3)$ into antitriplets and singlets with respect to the group $SU(3)$,

$$\tilde{\tau}^{81} := \frac{1}{2}(\tilde{S}^{9\,12} + \tilde{S}^{10\,11}), \qquad \tau^{82} := \frac{1}{2}(\tilde{S}^{9\,11} - \tilde{S}^{10\,12}),$$
$$\tau^{83} := \frac{1}{2}(\tilde{S}^{9\,14} + \tilde{S}^{10\,13}) \qquad \tau^{84} := \frac{1}{2}(\mathcal{S}^{9\,13} - \mathcal{S}^{10\,14}),$$
$$\tau^{85} := \frac{1}{2}(\tilde{S}^{11\,14} + \tilde{S}^{12\,13}), \qquad \tau^{86} := \frac{1}{2}(\tilde{S}^{11\,13} - \tilde{S}^{12\,14})$$

The spinorial representations of the group $SO(1, 13)$ are the direct product of polynomials of Tables I–IV. We can find all the members of a spinorial multiplet of the group $SO(1, 13)$ by applying $\tilde{S}^{ab}$ on any initial Grassmann odd product of polynomials if one polynomial is taken from Table I, another from Table II, and the third from Tables III and IV. In the same multiplet, there are triplets, singlets, and antitriplets with respect to $SU(3)$, which are doublets or singlets with respect to $SU(2)$ and are left- and right-handed with respect to $SO(1, 3)$.

5.2. Dynamical Arrangement of Representations of *SO*(1, 13) with Respect to Subgroups *SO*(1, 7) and *SO*(6)

To see how Yang–Mills fields enter into the theory, we shall rewrite the Weyl-like equation in the presence of the gravitational field (18) in terms of components of fields which determine gravitation in the four-dimensional subspace and of those which determine gravitation in higher dimensions, assuming that the coordinates of ordinary space with indices higher than four stay compacted to unmeasurably small dimensions (or cannot at all be noticed for some other reason). Since Grassmann space only manifests itself through average values of observables, compactification of a part of Grassmann space has in this sense no meaning. However, since parameters of Lorentz transformations in a freely falling coordinate system for both spaces have to be the same, no transformations to the fifth or higher coordinates may occur at measurable energies. Therefore, at low energies, the four-dimensional subspace of Grassmann space with the generators defining the Lorentz group $SO(1, 3)$ is (almost) decomposed from the rest of the Grassmann space with the generators forming the (compact) group $SO(d - 4)$ because of the decomposition of ordinary space. This is valid on the classical level only.

According to the previous subsection, the breaking of symmetry of $SO(1, 13)$ should, however, appear in steps, first through $SO(1, 7) \times SO(6)$ and later to the final symmetry, which is needed in the Standard Model for massless particles.

We shall comment on possible ways of spontaneously broken symmetries by studying the Weyl equation in the presence of gravitational fields in d dimensions for massless particles [Eqs. (17), (18)],

$$\tilde{\gamma}^a p_{0a} = 0, \qquad p_{0a} = f^{\mu}{}_{\alpha} p_{0\mu}, \qquad p_{0\mu} = p_{\mu} - \tfrac{1}{2}\tilde{S}^{ab}\omega_{ab\mu} \tag{20}$$

5.2.1. Standard Model

Before the breaking of the symmetry $SU(3) \times SU(2) \times U(1)$ into $SU(3) \times U(1)$, the canonical momentum $p_{0\alpha}$ ($\alpha = 0, 1, 2, 3$ and $d = 4$) includes the gauge fields connected with the groups $SU(3)$, $SU(2)$, and $U(1)$. We shall pay attention only to the groups $SU(2)$ and $U(1)$, which are involved in the breaking of symmetry,

$$p_{0\alpha} = p_{\alpha} - g\tau^i A^i{}_{\alpha} - g' Y B_{\alpha} \tag{21}$$

where g and g' are the two coupling constants. If $\tau^{\pm} = \tau^1 \pm i\tau^2$, then $A^{\pm}{}_{\alpha} = A^1{}_{\alpha} \mp iA^2{}_{\alpha}$. Let

$$A^3{}_{\alpha} = \frac{g/g'}{\sqrt{1 + (g/g')^2}} Z_{\alpha} + \frac{1}{\sqrt{1 + (g/g')^2}} A_{\alpha}$$

$$B_{\alpha} = -\frac{1}{\sqrt{1 + (g/g')^2}} Z_{\alpha} + \frac{g/g'}{\sqrt{1 + (g/g')^2}} A_{\alpha}$$

so that the transformation is orthonormalized, and one can easily rewrite Eq. (21) as follows:

$$p_{0\alpha} = p_{\alpha} - \frac{g}{2}(\tau^+ A^+{}_{\alpha} + \tau^- A^-{}_{\alpha}) + \frac{gg'}{\sqrt{g^2 + g'^2}} QA_{\alpha} + \frac{g^2}{\sqrt{g^2 + g'^2}} Q' Z_{\alpha}$$

$$Q = \tau^3 + Y, \qquad Q' = \tau^3 - \left(\frac{g'}{g}\right)^2 Y \tag{22}$$

In the Standard Model, $\langle Q \rangle$ is the conserved quantity and $\langle Q' \rangle = -\frac{1}{2}[1 + (g'/g)^2]$ is not, due to the fact that $\langle Q \rangle = 0$ for the Higgs fields in the ground state, while $\langle Q' \rangle \neq 0$.

If $g = g'$, then $Q = \tau^3 + Y$ and $Q' = \tau^3 - Y$. If no symmetry is spontaneusly broken, that is, if no Higgs breaks symmetry by making a choice for its ground-state symmetry, the only thing which has been done by introducing linear superpositions of fileds is the rearrangement of fields, which always can be done without any consequence, except that it may help to better see the symmetries.

Spontaneusly breaking of symmetries causes the nonconservation of quantum numbers as well as massive clusters of fields.

5.2.2. Spin Connections and Gauge Fields Leading to the Standard Model

We shall rewrite the canonical momentum of Eq. (20) to manifest possible ways of breaking symmetries of $SO(1, 13)$ down to the symmetries of the Standard Model. We first write

$$\tilde{\gamma}^a p_{0a} = 0 = \tilde{\gamma}^a f^{\mu}{}_a p_{0\mu} = (\tilde{\gamma}^m f^{\alpha}{}_m + \tilde{\gamma}^h f^{\alpha}{}_h) p_{0\alpha} + (\tilde{\gamma}^m f^{\sigma}{}_m + \tilde{\gamma}^h f^{\sigma}{}_h) p_{0\sigma} \tag{23}$$

with $\alpha, m \in \{0,1,2,3\}$ and $\sigma, h \in \{5,\ldots,14\}$ to separate the $(d = 4)$-dimensional subspace out of $(d = 14)$-dimensional space. We may further rearrange the canonical momentum $p_{0\mu}$,

$$p_{0\mu} = p_\mu - \tfrac{1}{2}\tilde{S}^{h_1 h_2}\omega_{h_1 h_2 \mu} - \tfrac{1}{2}\tilde{S}^{k_1 k_2}\omega_{k_1 k_2 \mu} - \tfrac{1}{2}\tilde{S}^{h_1 k_1}\omega_{h_1 k_1 \mu}$$

with $h_i \in \{0,1,\ldots,8\}$ and $k_i \in \{9,\ldots,14\}$, so that the $\tilde{S}^{h_1 h_2}$ define the algebra of the subgroup $SO(1,7)$, while the $\tilde{S}^{h_1 h_2}$ define the algebra of the subgroup $SO(6)$. The generators $\tilde{S}^{h_1 k_1}$ rotate states of a multiplet of the group $SO(1,13)$ into each other.

Taking into account Section 5.1, we may rewrite the generators $\tilde{S}^{ab}$ in terms of the corresponding generators of subgroups $\tilde{\tau}^{Ai}$, and accordingly, as in the Standard Model case, introduce new fields (see Section 5.2.1), which are superpositions of the old ones,

$$gA^{31}{}_\mu = \tfrac{1}{2}(\omega_{58\mu} - \omega_{67\mu}) \qquad gA^{32}{}_\mu = \tfrac{1}{2}(\omega_{57\mu} + \omega_{68\mu})$$

$$gA^{41}{}_\mu = \tfrac{1}{2}(\omega_{56\mu} + \omega_{78\mu}) \qquad gA^{32}{}_\mu = \tfrac{1}{2}(\omega_{56\mu} - \omega_{78\mu}) \tag{24}$$

$$gA^{51}{}_\mu = \tfrac{1}{2}(\omega_{58\mu} + \omega_{67\mu}), \qquad gA^{52}{}_\mu = \tfrac{1}{2}(\omega_{57\mu} - \omega_{68\mu}) \tag{25}$$

$$\tfrac{1}{2}\tilde{S}^{h_1 h_2}\omega_{h_1 h_2 \mu} = g\tilde{\tau}^{Ai}A^{Ai}{}_\mu \tag{26}$$

where for $A = 3$, $i = 1,2,3$, for $A = 4$, $i = 1$, and for $A = 5$, $i = 1,2$. Accordingly, the fields $A^{Ai}\mu$ are the gauge fields of the group $SU(2)$ if $A = 3$ and of $U(1)$ if $A = 4$. Since $\tilde{\tau}^{41}$ and $\tilde{\tau}^{5i}$ form the group $SU(2)$ as well, the corresponding fields could be the gauge fields of this group. The breaking of symmetry should make a choice between the gauge groups $U(1)$ and $SU(2)$.

We leave the notation for spin connection fields in the case that $h_i \in \{0,1,2,3\}$ unchanged. We also leave unchanged the spin connection fields for the case that $h_1 = 0,1,2,3$ and $h_2 = 5,6,7,8$ as well as for the case that $h_1 \in \{0,1,\ldots,8\}$ and $k_1 \in \{9,\ldots,14\}$, while we arrange terms with $k_i \in \{8,\ldots,14\}$ to demonstrate the symmetry $SU(3)$ and $U(1)$,

$$gA^{61}{}_\mu = \tfrac{1}{2}(\omega_{9\,12\mu} - \omega_{10\,11\mu}), \qquad gA^{62}{}_\mu = \tfrac{1}{2}(\omega_{9\,11\mu} + \omega_{10\,12\mu})$$

$$gA^{63}{}_\mu = \tfrac{1}{2}(\omega_{9\,10\mu} - \omega_{11\,12\mu}), \qquad gA^{64}{}_\mu = \tfrac{1}{2}(\omega_{9\,14\mu} - \omega_{10\,13\mu})$$

$$gA^{65}{}_\mu = \tfrac{1}{2}(\omega_{9\,13\mu} + \omega_{10\,14\mu}), \qquad gA^{66}{}_\mu = \tfrac{1}{2}(\omega_{11\,14\mu} - \omega_{12\,13\mu})$$

$$gA^{67}{}_\mu = \tfrac{1}{2}(\omega_{1113\mu} + \omega_{1214\mu}), \qquad gA^{68}{}_\mu = \frac{1}{2\sqrt{3}}(\omega_{910\mu} + \omega_{1112\mu} - 2\omega_{1314\mu}) \tag{27}$$

$$gA^{71}{}_{\mu} = -\tfrac{1}{2}(\omega_{9\,10\mu} + \omega_{11\,12\mu} + \omega_{13\,14\mu}) \tag{28}$$

We may accordingly define fields

$$gA^{81}{}_{\mu} = \tfrac{1}{2}(\omega_{9\,12\mu} + \omega_{10\,11\mu})$$
$$gA^{82}{}_{\mu} = \tfrac{1}{2}(\omega_{9\,11\mu} - \omega_{10\,12\mu})$$
$$gA^{83}{}_{\mu} = \tfrac{1}{2}(\omega_{9\,14\mu} + \omega_{10\,13\mu})$$
$$gA^{84}{}_{\mu} = \tfrac{1}{2}(\omega_{9\,13\mu} - \omega_{10\,14\mu})$$
$$gA^{85}{}_{\mu} = \tfrac{1}{2}(\omega_{11\,14\mu} + \omega_{12\,13\mu})$$
$$gA^{86}{}_{\mu} = \tfrac{1}{2}(\omega_{11\,13\mu} - \omega_{12\,14\mu})$$

so that it follows that

$$\tfrac{1}{2}\tilde{S}^{k_1k_2}\omega_{k_1k_2\mu} = g\tilde{\tau}^{Ai}A^{Ai}{}_{\mu}$$

with $A = 6, 7, 8$. While $A^{6i}{}_{\mu}$, $i \in \{1, \ldots, 8\}$, form the gauge field of the group $SU(3)$ and $A^{71}{}_{\mu}$ corresponds to the gauge group $U(1)$, terms $g\tilde{\tau}^{7i}\,A^{7i}{}_{\mu}$ transform $SU(3)$ triplets into singlets and antitriplets. Again, without additional requirements, all the coupling constants g are equal. To be in agreement with what the Standard Model needs as an input, we further rearrange the gauge fields belonging to the two $U(1)$ fields, one coming from the subgroup $SO(1, 7)$, the other from the subgroup $SO(6)$. We therefore define

$$Y_1 = (\tau^{41} + \tau^{71}), \qquad Y_2 = -\,(\tau^{41} - \tau^{71})$$

and similarly to the Standard Model case of Section 5.2.1,

$$A^{1}{}_{\mu} = \tfrac{1}{2}(A^{41}{}_{\mu} + A^{71}{}_{\mu}), \qquad A^{2}{}_{\mu} = -\tfrac{1}{2}(A^{41}{}_{\mu} - A^{71}{}_{\mu}) \tag{29}$$

The rearrangement of fields demonstrates all the symmetries of the massless particles of the Standard Model and more. Taking into account Tables I–III, one finds for the quantum numbers of spinors, which belong to a multiplet of $SO(1, 7)$ with left-handed $SU(2)$ doublets and right-handed $SU(2)$ singlets and which are triplets or singlets with respect to $SU(3)$, the ones presented on Table IV. We use the names of the Standard Model to denote triplets and singlets with respect to $SU(3)$ and $SU(2)$.

Table V gives the expectation values for the generators $\tilde{\tau}^{63}$ and $\tilde{\tau}^{68}$ of the group $SU(3)$ and the generator $\tilde{\tau}^{71}$ of the group $U(1)$ [the two groups are subgroups of the group $SO(6)$] and of the generators $\tilde{\tau}^{33}$ of the group $SU(2)$, $\tilde{\tau}^{41}$ of the group $U(1)$ and $\tilde{\Gamma}^{(4)}$ of the group $SO(1, 3)$ [the three groups are subgroups of the group $SO(1, 7)$] for the multiplet [with respect to $SO(1, 7)$] which contains left-handed ($\langle\Gamma^{(4)}\rangle = -1$) $SU(2)$ doublets and right-handed

Table V. Expectation Values for the Generators of the Group $SU(3) \times U(1) \subset SO(6)$ and of the Generators of the Group $SU(2) \times U(1) \times SO(1,3) \subset SO(1,7)$

		$SU(2)$ doublets					
		$\tilde{\tau}^{33}$	$\tilde{\tau}^{41}$	$\tilde{\tau}^{71}$	$\tilde{Y}_1$	$\tilde{Y}_2$	$\Gamma^{(4)}$
$SU(3)$ triplets							
$\tilde{\tau}^{6\,3} = (\frac{1}{2}, -\frac{1}{2}, 0)$	u_i	1/2	0	1/6	1/6	1/6	−1
$\tilde{\tau}^{6\,8} = \left(\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$	d_i	−1/2	0	1/6	1/6	1/6	−1
$SU(3)$ singlets							
$\tilde{\tau}^{6\,3} = 0$	ν_i	1/2	0	−1/2	−1/2	−1/2	−1
$\tilde{\tau}^{6\,8} = 0$	e_i	−1/2	0	−1/2	−1	−1	−1
		$SU(2)$ singlets					
		$\tilde{\tau}^{33}$	$\tilde{\tau}^{41}$	$\tilde{\tau}^{71}$	$\tilde{Y}_1$	$\tilde{Y}_2$	$\tilde{\Gamma}^{(4)}$
$SU(3)$ triplets							
$\tilde{\tau}^{6\,3} = (\frac{1}{2}, -\frac{1}{2}, 0)$	u_i	0	1/2	1/6	2/3	−1/3	1
$\tilde{\tau}^{6\,8} = \left(\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$	d_i	0	−1/2	1/6	−1/3	2/3	1
$SU(3)$ singlets							
$\tilde{\tau}^{6\,3} = 0$	ν_i	0	1/2	−1/2	0	−1	1
$\tilde{\tau}^{6\,8} = 0$	e_i	0	−1/2	−1/2	−1	0	1

($\langle \Gamma^{(4)} \rangle = 1$) $SU(2)$ singlets. The index i of u_i, d_i, ν_i and e_i runs over the four families presented in Table I.

We see that, besides $\tilde{Y}_2$, these are just the quantum numbers needed for massless fermions of the Standard Model. The value for the additional hypercharge $\tilde{Y}_2$ is nonzero for the right-handed neutrinos as well as for other states, except for right-handed electrons.

Since no symmetry is broken yet, all the gauge fields are of the same strength. To come to the symmetries of massless fields of the Standard Model, surplus symmetries should be broken so that all the coupling constants connected with the fields $\omega_{ab\mu}$ which do not determine the fields $A^{Ai}{}_{\mu}$, $A = 3, 6$ [Eqs. (24), (27)] and A^1_μ [Eq. (29)] should be small and yet the coupling constants of these three fields should not be equal. Accordingly also the operator $\tilde{Y}_2$ could depend, similarly to the case of Eq. (22), on the coupling constants.

The mirror symmetry should be broken so that multiplets of $SO(1,7)$ with right-handed $SU(2)$ doublets and left-handed $SU(2)$ singlets become very massive. All the surplus multiplets, either bosonic or fermionic, should become of large enough masses not to be measurable yet. The proposed approach predicts four rather than three families of fermions. Although in this paper, we do not discuss possible ways of appearance of spontaneously broken symmetries, bringing the symmetries of the group $SO(1, 13)$ down to symmetries of the Standard Model, we still would like to know whether there are terms in the Weyl equation (23) which may behave like the Yukawa couplings. We see that indeed the term $\tilde{\gamma}^h f^{\sigma}{}_h p_{0\sigma}$ with $h \in \{5, 6, 7, 8\}$ and $\sigma \in \{5, 6, \ldots\}$ really may, if operating on a right-handed $SU(2)$ singlet, transform it to a left-handed $SU(2)$ doublet. We also can find among scalars the terms with quantum numbers of Higgs bosons which are $SU(2)$ doublets with respect to operators of vectorial character.

6. CONCLUDING REMARKS

In this paper, we demonstrated that, if we assume that the space has d commuting and d anticommuting coordinates, then, for $d \geq 14$, all spins in d dimensions described in the vector space spanned over the space of anticommuting coordinates manifest in four-dimensional subspace as the spins and all the charges, unifying spins and charges of fermions and bosons independently, although the supersymmetry, which guarantees the same number of fermions and bosons, is a manifesting symmetry. The anticommuting coordinates can be represented by either Grassmann coordinates or the Kähler differential forms.

We demonstrated that either our approach or the approach of differential forms suggest four families of quarks and leptons, rather than three.

We have shown that starting (in any of the two approaches) with the Lorentz symmetry in the tangent space in $d \geq 14$, spin degrees of freedom (described by dynamics in the space of anticommuting coordinates) manifest in four-dimensional subspace as spins and color, weak and hypercharges, with one additional hypercharge, in such a way that only left-handed weak charge doublets together with right-handed weak charge singlets appear, if the symmetry is spontaneously broken from $SO(1, 13)$ first to $SO(1, 7)$ and $SO(6)$, so that a multiplet of $SO(1, 7)$ with only left-handed $SU(2)$ doublets and right-handed $SU(2)$ singlets survive, while the mirror symmetry is broken, and then to $SO(1, 3)$, $SU(2)$, $SU(3)$, and $U(1)$.

We have demonstrated that the gravity in d dimensions manifests as ordinary gravity and all gauge fields in four-dimensional subspace, after the breaking of symmetry and the accordingly changed coupling constant. We also have shown that there are terms in the Weyl equations which in four-dimensional subspace manifest as Yukawa couplings.

The two approaches, the Kähler one after the generalization which we have suggested, and ours, lead to the same results.

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REFERENCES

Becher, P., and H. Joos (1982). The Dirac–Kähler equation and fermions on the lattice, *Z. Phys. C—Particles Fields*, **15**, 343–365.

Borštnik, Anamarija, and Norma Susana Mankoč Borštnik (1999). Are spins and charges unified? How can one otherwise understand connection between handedness (spin) and weak charge? in: N. Mankoč Borštnik, H. B. Nielsen, and C. Froggatt, Editors, *Proceedings of the International Workshop What Comes Beyond the Standard Model, Bled, Slovenia, 1998*, DMFA, Založništvo, p. 52–57; hep-ph/9905357.

Georgi, Howard (1982). *Lie Algebra in Particle Physics*, Benjamin/Cummings, Menlo Park, California.

Ikemori, Hitochi (1987). Superfield formulation of relativistic superparticle, *Physics Letters*, **199**, 239–242.

Káhler, Erich (1962). Der innere Differentialkalkül, *Rend. Mat. Ser. V*, **21**, 452–523.

Mankoč Borštnik, Norma Susana (1992a). Spin connection as a superpartner of a vielbein, *Physics Letters B*, **292**, 25–29.

Mankoč, Borštnik, Norma Susana (1992b). From a world-sheet supersymmetry to the Dirac equation, *Nuovo Cimento A*, **105**, 1461–1471.

Mankoč Borštnik, Norma Susana (1993). Spinor and vector representations in four dimensional Grassmann space, *Journal of Mathematical Physics*, **34**, 3731–3745.

Mankoč Borštnik, Norma Susana (1994a). Spinors, vectors and scalars in Grassmann space and canonical quantization for fermions and bosons, *International Journal of Modern Physics A*, **9**, 1731–1745.

Mankoč Borštnik, Norma Susana (1994b). Unification of spins and charges in Grassmann space, hep-th/9408002.

Mankoč Borštnik, Norma Susana (1994c). Qantum mechanics in Grassmann space, supersymmetry and gravity, hep-th/9406083.

Mankoč Borštnik, Norma Susana (1995a). Poincaré algebra in ordinary and Grassmann space and supersymmetry, *Journal of Mathematical Physics*, **36**,1593–1601.

Mankoč Borštnik, Norma Susana (1995b). Unification of spins and charges in Grassmann space, *Modern Physics Letters A*, **10**, 587–595; hep-th/9512050.

Mankoč Borštnik, Norma Susana (1999). Unification of spins and charges in Grassmann space, in: N. Mankoč Borštnik, H. B. Nielsen, and C. Froggatt, Editors, *Proceedings to the International Workshop What Comes Beyond the Standard Model Bled, Slovenia, 1998*, DMFA, Založništvo, pp. 20–29; hep-ph/9905357.

Mankoč Borštnik, Norma Susana, and Svjetlana Fajfer (1997). Spins and charges, the algebra and subalgebras of the group SO(1,14), *Nuovo Cimento B*, **112**, 1637–1665; hep-th/9506175.

Mankoč Borštnik, Norma Susana, and Holger Bech Nielsen (1999). Dirac–Kähler approach conencted to quantum mechanics in Grassmann space, *Physical Review D*, **15**; in N. Mankoč Borštnik, H. B. Nielsen, C. Froggatt, and Editors, *Proceedings to the International Workshop What Comes Beyond the Standard Model, Bled, Slovenia, 1998,* DMFA Založništvo, pp. 68–73; hep-ph/9905357; hep-th/9909169; hep-th/9911032.

Nielsen, Holger Bech, and M. Ninomija (1981a). A no-go theorem for regularizing chiral fermions, *Physics Letters B*, **105**, 219–223.

Nielsen, Holger Bech, and M. Ninomija (1981b). Absence of neutrinos on a lattice, *Nuclear Physics B*, **185**, 20–40.

Wess, Julius, and Jonathan Bagger (1983). *Supersymmetry and Supergravity*, Prince University Press, Princeton, New Jersey.